Functions

What is a Function?

To provide the classical understanding of functions, think of a *function* as a kind of machine. You feed the machine raw materials, and the machine changes the raw materials into a finished product based on a specific set of instructions. The kinds of functions we consider here, for the most part, take in a real number, change it in a formulaic way, and give out a real number. Think of this as an *input-output machine*; you give the function an input, and it gives you an output.

Example: A function.

$$f(x) = x^2$$

So if we look at the squaring function, $f(x) = x^2$, that's our machine. We put in x, and we get out f(x). If we put in x as 4, the function defines f(x) as 4^2, or 16. If x is 1, then f(x) is 1^2, which is also 1. If x is -5, f(x) is 25. A function always performs the same action on its input.

A function is usually written as *f*, *g*, or something similar - although it doesn't have to be. A function is always defined as "of a variable" which tells us what to replace in the formula for the function.

For example, $f(x) = 3x + 2$ tells us:

- The function *f* is a function of *x*. This means that x is the input value.
- To evaluate the function at a certain number, replace the *x* with that number. That number is now the input value.
- Replacing *x* with that number in the right side of the function will produce the function's output for that certain input.

Thus, if we want to know the value (or output) of the function at 3:

$$f(x) = 3x + 2$$

$$f(3) = 3(3) + 2$$ We evaluate the function at *x* = 3 by plugging in 3 for x.

$$f(3) = 9 + 2 = 11$$ The value of f at 3 is 11. So, if 3 is the input, 11 is the output.

Note that $f(3)$ means the value of the dependent variable—the output—when x--the input—takes on the value of 3. So we see that the number *11* is the output of the function when we give the number *3* as the input. We refer to the input as the **argument** of the function (or the **independent variable**), and to the output as the **value** of the function at the given argument (or the **dependent variable**). A good way to think of it is the dependent variable $f(x)$ 'depends' on

the value of the independent variable x. This is read as "the value of *f* at three is eleven", or simply "*f* of three equals eleven".

Notation

Functions are used so much that there is a special notation for them. The notation is somewhat unclear, so familiarity with it is important in order to understand the meaning of an equation or formula.

Though there are no strict rules for naming a function, it is standard practice to use the letters *f*, *g*, and *h* to denote functions (you might see f(x), g(x) or h(x)), and the variable *x* to denote an independent variable. *y* is used for both dependent and independent variables. For example, you sometimes see a function written "$y = x^2$"; this is the same as saying "$f(x) = x^2$".

When discussing or working with a function *f*, it's important to know not only the function, but also its independent variable *x*. To find the output of the machine, you have to know what you're putting in and what the machine will do to it. Thus, when referring to a function *f*, you usually do not write *f*, but instead *f*(*x*). The function is now referred to as "*f* of *x*". The name of the function, f, is adjacent to the independent variable, x (in parentheses). This is useful for indicating the value of the function at a particular value of the independent variable—which is the dependent variable. For instance, if

$$f(x) = 7x + 1,$$

and if we want to use the value of *f* for *x* equal to 2, then we would substitute 2 for *x* on both sides of the definition above and write

$$f(2) = 7(2) + 1 = 14 + 1 = 15$$

As you can see, "f" represents the function, "x" is the independent variable and "f(x)" represents the dependent variable. This notation is more informative than leaving off the independent variable and writing simply "f" or "y," because it shows that the dependent variable is directly related to the independent variable, but it can be ambiguous since the parentheses can be misinterpreted as multiplication.

Modern understanding of functions: domain and range

The formal definition of a function states that a function is actually a *rule* that associates elements of one set called the *domain* of the function, with the elements of another set called the *range* of the function. For each value we select from the domain of the function, there exists exactly one corresponding element in the range of the function. The definition of the function tells us which element in the range corresponds to the element we picked from the domain. Classically, the element picked from the domain is pictured as something that is fed into the function and the corresponding element in the range is pictured as the output. Since we "pick"

the element in the domain whose corresponding element in the range we want to find, we have control over what element we pick and hence this element is also known as the "independent variable". The element mapped in the range is beyond our control and is "mapped to" by the function. This element is hence also known as the "dependent variable", for it depends on which independent variable we pick. Since the elementary idea of functions is better understood from the classical viewpoint instead of this modern one, we shall use the classical hereafter. However, it is still important to remember the correct definition of functions at all times.

To make it simple, for the function *f*(*x*), all of the possible *x* values constitute the domain, and all of the values *f*(*x*) (*y* on the x-y plane) constitute the range.

In other words:

Each function has a set of values, the function's *domain*, which it can accept as input. Perhaps this set is all positive real numbers; perhaps it is the set {pork, mutton, beef}. This set must be implicitly/explicitly defined in the definition of the function. You cannot feed the function an element that isn't in the domain, as the function is not defined for that input element.

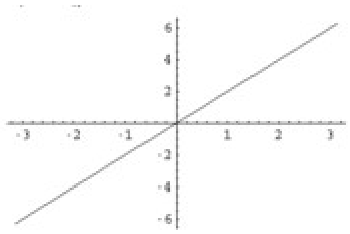
For example, let’s say our function is y= 2/x. This function is defined for every number except 0, because 2/0 does not exist, since you can’t divide by 0. So the domain of the function is every real number except 0.

Or, take the set {pork, mutton, beef}. Maybe our function is to marinate the input. You can marinate pork, mutton, and beef, but you can’t marinate a tennis ball. A tennis ball is not in the domain of the function, because the function can’t be performed on that input.

Each function has a set of values, the function's *range*, which it can output. This may be the set of real numbers. It may be the set of positive integers or even the set {0,1}. This set, too, must be implicitly/explicitly defined in the definition of the function.

Let’s take again $y=x^2$. The domain of this function is all real numbers, but the range is all positive real numbers and 0, because nothing can be squared to give a negative number. It doesn’t matter what the input is, the output will never be a negative number, so negative numbers are not in the range.

Graphing Functions



Graph of y=2x

It is sometimes difficult to understand the behavior of a function given only its definition; a visual representation or graph can be very helpful. A **graph** is a set of points in the Cartesian plane, where each point (x,y) indicates that $f(x) = y$. In other words, a graph uses the position of a point in one direction (the *vertical-axis* or *y-axis*) to indicate the value of *f* for a position of the point in the other direction (the *horizontal-axis* or *x-axis*).

Functions may be graphed by finding the value of *f* for various *x* and plotting the points (x, $f(x)$) in a Cartesian plane. So, the x-coordinate is the input, and the y-coordinate is the output. For the functions that you will deal with, the parts of the function between the points can generally be approximated by drawing a line or curve between the points. Extending the function beyond the set of points is also possible, but becomes increasingly inaccurate.

Plotting points like this is laborious. Fortunately, many functions' graphs fall into general patterns. For a simple case, consider functions of the form

$$f(x) = ax$$

The graph of *f* is a single line, passing through (0,0) and (1,a)—if x =0, then f(x) = a(0) = 0. If x = 1, then f(x) = a(1) = a. Thus, after plotting the two points, a straightedge may be used to draw the graph as far as is needed. After having learned calculus, you will know many more techniques for drawing good graphs of functions.

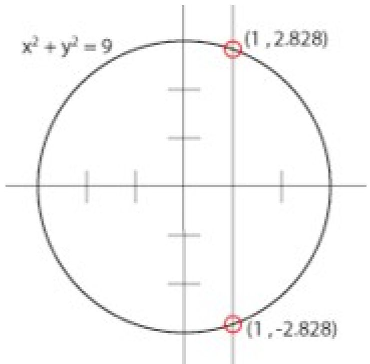
The Vertical Line Test: How to spot a non-function in disguise

By definition, for each "input" a function returns only one "output", corresponding to that input. While the same output may correspond to more than one input, one input cannot correspond to more than one output. So if you put one thing into the machine, you'll only get one thing out. Sometimes, however, if you put two different things into the machine, you'll get the same thing out each time.

For example: Let's take the squaring function again: $f(x) = x^2$.

If we put in 2 for x, we get out $2^2=4$ for f(x). We won't ever get anything else but 4. But if we put in -2 for x, we get out $(-2)^2=4$ for f(x). So by putting in two different inputs, we get the same output each time, but we will never get more than one output for a single input. That's what makes $f(x)=x^2$ a function.

This is expressed graphically as the *vertical line test*: a line drawn parallel to the axis of the dependent variable (normally vertical, since the axis of the dependent variable is usually the x-axis) will intersect the graph of a function only once. However, a line drawn parallel to the axis of the independent variable (normally horizontal) may intersect the graph of a function as many times as it likes. Equivalently, this has an algebraic (or formula-based) interpretation. We can always say if $a = b$, then $f(a) = f(b)$, but if we only know that $f(a) = f(b)$ then we can't be sure that $a = b$. This is because in the first case a and b are the inputs, and we know that the same input will always give the same output, so if we know that the inputs are the same, we know the outputs will be the same. In the second case, f(a) and f(b) are the outputs, and since two different inputs can give the same output, the fact that the outputs are the same does not necessarily mean that the inputs are the same.



This picture is an example of an expression which fails the vertical line test. If you put in 1 as the input, you get out two outputs, 2.828 and -2.828. The graph shows that $x^2 + y^2 = 9$ is not a function.

One-to-one Functions

A function $f(x)$ is **one-to-one** (or less commonly **injective**) if, for every value of f, there is only one value of x that corresponds to that value of f. For instance, the function $f(x) = \sqrt{1 - x^2}$ is not one-to-one, because both $x = 1$ and $x = -1$ result in $f(x) = 0$. However, the function $f(x) = x + 2$ is one-to-one, because, for every possible value of $f(x)$, there is exactly one corresponding value of x. Other examples of one-to-one functions are $f(x) = x^3 + ax$, where $a \in [0, \infty)$. Note that if you have a one-to-one function and translate or dilate it, it remains one-to-one. (As long as you don't multiply x or f by zero).

If you know what the graph of a function looks like, it is easy to determine whether or not the function is one-to-one. If every horizontal line intersects the graph in at most one point, then the function is one-to-one. This is known as the Horizontal Line Test. So, a one-to-one function passes both the vertical and horizontal line tests.

Important kinds of functions

Constant function

$$f(x) = c$$

It disregards the input and always outputs the constant c, and is a polynomial of the *zeroth* degree where $f(x) = cx^0 = c(1) = c$. Its graph is a horizontal line. Anything to the zeroth degree is one, so since x in this function is not seen on the right hand side, we can say that the constant is being multiplied by x^0, which is 1. Thus, it doesn't matter what x is, the output is always the constant.

An example of this type of function is the function $f(x) = 5$. No matter what the input is, the output is always 5.

Linear function

$$f(x) = mx + c$$

Takes an input, multiplies by *m* and adds *c*. It is a polynomial of the *first* degree, because m is multiplied by x^1, which is simply x. Its graph is a line (slanted, except *m* = 0).

An example of this type of function is f(x) = 3x + 2. 3 is *m*, and 2 is *c*. Note that if *m* is 0, the function becomes a constant function.

Identity function

$$f(x) = x$$

Takes an input and outputs it unchanged. A polynomial of the *first* degree, $f(x) = x^1 = x$. Special case of a linear function.

This function looks like a straight line at a 45-degree angle sloping up from left to right.

Quadratic function

$$f(x) = ax^2 + bx + c$$

A polynomial of the *second* degree, because the **highest power** of x present is x^2. Its graph is a parabola, unless *a* = 0. (Don't worry if you don't know what this is.)

An example of this type of function is $4x^2 + 2x + 3$. Again, if *a* is 0, the function becomes a linear function.

Polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

The number *n* is called the *degree*. This is a representation of a function with a degree higher than 2. Instead of *a*, *b, c,* etc, the coefficients are represented as a_n, a_{n-1}, etc. This makes things simpler when there are many terms in the function.

Let's take for example the function $f(x)=6x^5 + 3x^4 + x^2 + 4x + 7$. The degree, or *n,* is 5, because 5 is the highest degree of x present. Thus, 4 is *n-1*, etc. 6 is a_n, 3 is a_{n-1}, etc. Note that we do not see an x^3 term. This simply means that a_{n-2} is 0. In the same way, a_{n-3} or a_2 is 1, which is why we don't see a number next to x^2.

Signum function

$$\operatorname{sgn}(x) = \begin{cases} -1 & : \quad x < 0 \\ 0 & : \quad x = 0 \\ 1 & : \quad x > 0. \end{cases}$$

Determines the sign of the argument *x*—whether *x* is positive, negative or 0.

Example functions

Some more simple examples of functions have been listed below.

$$h(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$$ Gives 1 if input is positive, -1 if input is negative. Note that the function only accepts negative and positive numbers, not 0. Mathematics describes this condition by saying 0 is not in the domain of the function.
$$g(y) = y^2$$ Takes an input and squares it. $$g(z) = z^2$$ Exactly the same function, rewritten with a different independent variable—*z* instead of *y*. This is perfectly legal and sometimes done to prevent confusion (e.g. when there are already too many uses of *x* or *y* in the same paragraph.)
$$f(x) = \begin{cases} 5^{x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$ Note that we can define a function by a totally arbitrary rule. There may be no real life scenario in which the output is 5 to the power of the input squared if the input is positive, and zero if the input is negative or zero. It doesn't matter, though; we can still write a function for it.

Manipulating functions: What am I supposed to do with this thing?

Arithmetic with Functions

Functions can be manipulated in the same ways as variables; they can be added, multiplied, raised to powers, etc. For instance, let

$$f(x) = 3x + 2$$

$$g(x) = x^2.$$

Then

$$\begin{aligned} f + g &= (f+g)(x) \\ &= f(x) + g(x) \\ &= (3x+2) + (x^2) \\ &= x^2 + 3x + 2 \end{aligned},$$

Note that (f + g)(x) is not (f + g) multiplied by x. It is (f + g) *of* x, but it can still be written f(x) + g(x).

$$\begin{aligned} f - g &= (f - g)(x) \\ &= f(x) - g(x) \\ &= (3x + 2) - (x^2) \\ &= -x^2 + 3x + 2 \end{aligned}$$,

$$\begin{aligned} f \times g &= (f \times g)(x) \\ &= f(x) \times g(x) \\ &= (3x + 2) \times (x^2) \\ &= 3x^3 + 2x^2 \end{aligned}$$,

Here it is especially important that (f x g) of x not be confused with (f x g) multiplied by x. This is because, unlike addition and subtraction, multiplication is *not* distributive, meaning that you cannot write (f x g) multiplied by x as f(x) x g(x). You can, however write (f x g) of x that way.

$$\begin{aligned} \frac{f}{g} &= \left(\frac{f}{g}\right)(x) \\ &= \frac{f(x)}{g(x)} \\ &= \frac{3x + 2}{x^2} \\ &= \frac{3}{x} + \frac{2}{x^2} \end{aligned}$$.

Composition of functions

However, there is one particular way to combine functions which cannot be done with variables. The value of a function *f,* the output, depends upon the value of another variable *x,* the input; however, that variable could be equal to another function *g*, so its value depends on the value of a third variable. So the input of the function *f* is the output of the function *g,* so to know the output of the function *f,* we need to know the input of the second function, *g.* If this is the case, then the first variable, the output of *f*, is a function *h* of the third variable, the input of *g*.; this function (*h*) is called the **composition** of the other two functions (*f* and *g*). Composition is denoted by

$$f \circ g = (f \circ g)(x) = f(g(x)).$$

This can be read as either "f composed with g" or "f of g of x." This represents the function *h* which will get us from the input of *g* to the output of *f*.
For instance, let

$$f(x) = 3x + 2$$ and

$$g(x) = x^2.$$

Then

$$\begin{aligned} h(x) &= f(g(x)) \\ &= f(x^2) \\ &= 3(x^2) + 2 \\ &= 3x^2 + 2 \end{aligned}.$$

So we basically plug g(x) in for x in function f.

Here, *h* is the composition of *f* and *g* and we write $h = f \circ g$. Note that composition is not commutative:

$$f(g(x)) = 3x^2 + 2,$$ and

$$\begin{aligned} g(f(x)) &= g(3x+2) \\ &= (3x+2)^2 \\ &= 9x^2 + 12x + 4 \end{aligned}$$

so $f(g(x)) \neq g(f(x))$. Plugging g(x) in for x in f is not the same as plugging f(x) in for x in g. It matters what order you do the functions in.

Composition of functions is very common, mainly because functions themselves are common.

Transformations

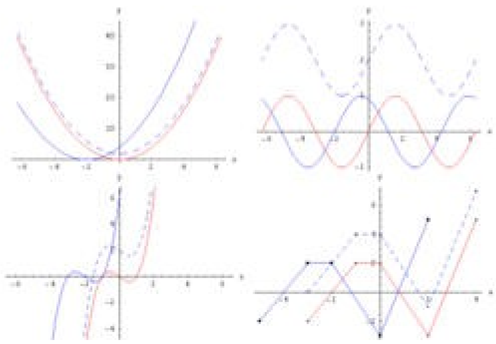
Transformations are a type of function manipulation that is very common. They consist of multiplying, dividing, adding or subtracting constants to either the input or the output. Multiplying by a constant is called **dilation,** or stretching/shrinking, and adding a constant is called **translation,** or shifting. Here are a few examples:

$f(2 \times x)$ Dilation

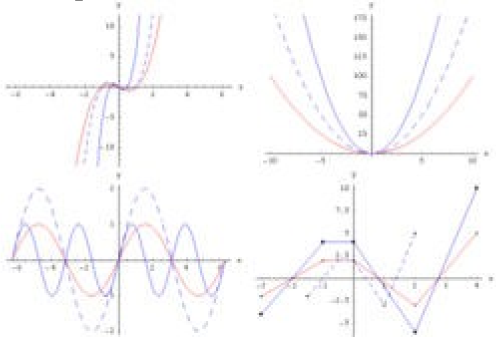
$f(x + 2)$ Translation

$2 \times f(x)$ Dilation

$2 + f(x)$ Translation



Examples of horizontal and vertical translations



Examples of horizontal and vertical dilations

Translations and dilations can be either horizontal or vertical. Examples of both vertical and horizontal translations can be seen above. The red graphs represent functions in their 'original' state, the solid blue graphs have been translated (shifted) horizontally, and the dashed graphs have been translated vertically.

Dilations are demonstrated in a similar fashion. The function

$$f(2 \times x)$$

has had its input doubled. One way to think about this is that now any change in the input will be doubled. If I add one to x, I add two to the input of f, so the input of f will now change twice as quickly, meaning the output will also change more quickly. Thus, this is a horizontal dilation by

$\frac{1}{2}$ because the distance to the *y*-axis has been **halved**. In effect, you have made the graph of the function 'narrower,' because to get to a certain height on the y-axis you don't have to go as far on the x-axis, since the input is being doubled each time. That's why even though you're multiplying by two, it's a horizontal dilation by one half. A vertical dilation, such as

$$2 \times f(x)$$

is slightly more straightforward. In this case, you double the output of the function. The output represents the distance from the *x*-axis, so in effect, you have made the graph of the function 'taller'. So a horizontal dilation by one half and a vertical dilation by two of a certain function will have the same graph. Here are a few basic examples where *a* is any positive constant:

Original graph	$f(x)$	Reflection about origin—this flips the graph across both axes.	$-f(-x)$
Horizontal translation by *a* units **left**—note that the same way horizontal dilations are "backwards" in that multiplying the input by two gives a horizontal dilation of ½, horizontal translations are "backwards" because adding to the input moves the graph in the negative direction, left.	$f(x+a)$	Horizontal translation by *a* units **right**—note that subtracting from the input moves the graph in the positive direction, right. Horizontal translations are "backwards" like this because, say if you subtract two from the input, to get the same output you have to start with an input that is two more, since you will then subtract two from it. Thus subtracting from the input moves the graph to the right, in the positive direction.	$f(x-a)$
Horizontal dilation by a factor of *a*—multiply the input by the reciprocal of a.	$f(x \times \frac{1}{a})$	Vertical dilation by a factor of *a*—multiply the output by a.	$a \times f(x)$
Vertical translation by *a* units **down**	$f(x) - a$	Vertical translation by *a* units **up**	$f(x) + a$
Reflection about *x*-axis	$-f(x)$	Reflection about *y*-axis	$f(-x)$

Summary of Dilations and Translations: Dilation means multiply. Translation means add or subtract. Horizontal means multiply or add to the input, or only the x. Vertical means multiply or add to the output, or the entire f(x). Horizontal dilations and translations are "backwards."

Inverse functions

We call $g(x)$ the inverse function of $f(x)$ if, for all x:

$$g(f(x)) = f(g(x)) = x$$

. So, the two functions applied one after the other cancel each other out, and the output is the same input that we started with.

A function $f(x)$ has an inverse function if and only if $f(x)$ is one-to-one. For example, the inverse of $f(x) = x + 2$ is $g(x) = x - 2$. The function $f(x) = \sqrt{1 - x^2}$ has no inverse.

Notation

The inverse function of f is denoted as $f^{-1}(x)$. Thus, $f^{-1}(x)$ is defined as the function that follows this rule:

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

To determine $f^{-1}(x)$ when given a function f, substitute $f^{-1}(x)$ for x and substitute x for $f(x)$. Then solve for $f^{-1}(x)$, provided that it is also a function.

Example: Given $f(x) = 2x - 7$, find $f^{-1}(x)$.

Substitute $f^{-1}(x)$ for x and substitute x for $f(x)$. Then solve for $f^{-1}(x)$:

$$f(x) = 2x - 7$$

$$x = 2[f^{-1}(x)] - 7$$

$$x + 7 = 2[f^{-1}(x)]$$

$$\frac{x+7}{2} = f^{-1}(x)$$

To check your work, confirm that $f^{-1}(f(x)) = x$:

$f^{-1}(f(x)) =$

$f^{-1}(2x - 7) =$

$$\frac{2x - 7 + 7}{2} = \frac{2x}{2} = x$$

If f isn't one-to-one, then, as we said before, it doesn't have an inverse. Then this method will fail.

Example: Given $f(x) = x^2$, find $f^{-1}(x)$.

Substitute $f^{-1}(x)$ for x and substitute x for $f(x)$. Then solve for $f^{-1}(x)$:

$$f(x) = x^2$$

$$x = (f^{-1}(x))^2$$

$$f^{-1}(x) = \pm\sqrt{x}$$

Since there are two possibilities for $f^{-1}(x)$, it's not a function. Thus $f(x) = x^2$ doesn't have an inverse. Of course, we could also have found this out from the graph by applying the Horizontal Line Test. It's useful, though, to have lots of ways to solve a problem, since in a specific case some of them might be very difficult while others might be easy. For example, we might only know an algebraic expression for $f(x)$ but not a graph.

Interval Notation:

Remember way up there, where we talked about domain and range? We're going to talk about that again. The domain and range of functions are commonly expressed using interval notation. This notation is very simple, but sometimes ambiguous because of the similarity to ordered pair notation:

Meaning	Interval Notation	Set Notation
All values greater than or equal to *a* and less than or equal to *b*	$[a, b]$	$\{x : a \leq x \leq b\}$
All values greater than *a* and less than *b*	(a, b)	$\{x : a < x < b\}$
All values greater than or equal to *a* and less than *b*	$[a, b)$	$\{x : a \leq x < b\}$
All values greater than *a* and less than or equal to *b*	$(a, b]$	$\{x : a < x \leq b\}$
All values greater than or equal to *a*.	$[a, \infty)$	$\{x : x \geq a\}$
All values greater than *a*.	(a, ∞)	$\{x : x > a\}$
All values less than or equal to *a*.	$(-\infty, a]$	$\{x : x \leq a\}$
All values less than *a*.	$(-\infty, a)$	$\{x : x < a\}$
All values.	$(-\infty, \infty)$	$\{x : x \in \mathbb{R}\}$

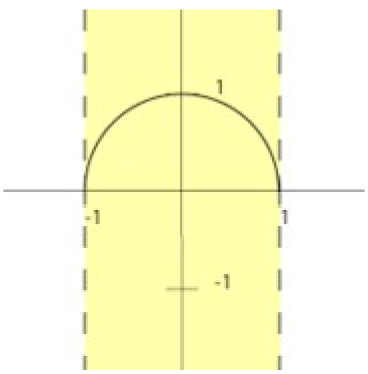
As you can see, a bracket, [ ], means that the value shown is included in the domain, whereas a parenthesis, ( ), means that the value shown is not included. Thus, "[a, b]" means "all values

greater than *or equal to* a and less than *or equal to* b", or all values between a and b, including a and b, whereas "(a, b)" means "all values greater than a and less than b", or all values between a and b, *not including* a and b.

Note that ∞ and $-\infty$ must always have an exclusive parenthesis rather than an inclusive bracket. This is because ∞ is not a number, and therefore cannot be in our set. So if the domain is all real numbers, the interval is written (-∞, ∞) to mean all numbers between negative infinity and infinity, but not including negative infinity and infinity, because those are not real numbers. ∞ is really just a symbol that makes things easier to write, like the intervals above.

So, take for example the function f(x) = √x. We know that the domain of this function is all positive numbers and zero. We know this because any number squared becomes positive. Since it is not possible to square a number and get a negative result, it is not possible to take the square root of a negative number, since this is in effect "un-squaring" the number. √-1 does not exist because nothing squared is -1. Zero, however, is included, because 0^2 is 0, so √0 is 0. Thus, we write the domain of the function in interval form, [0, ∞). We have a bracket next to 0 because 0 is part of the domain, and a parenthesis next to ∞ because ∞ is not part of the domain.

Example: Domain, Range and Interval Notation

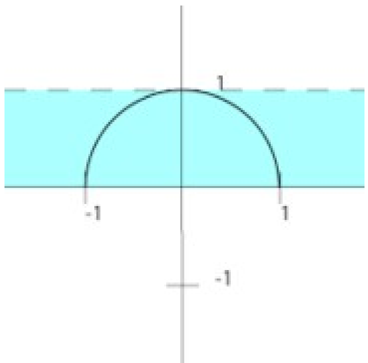


This picture shows the graph of the function

$$f(x) = \sqrt{1 - x^2}$$

The domain of this function is written in interval notation, $[-1, 1]$. X cannot be bigger than 1 or smaller than -1, because if it were, x^2 would be bigger than one, and then we would have to take the square root of a negative number, which is not possible. We use brackets because -1 and 1 are included in the domain, but nothing beyond. In other words,

$f(x)$is defined for $x \in [-1, 1]$, or$\{x : -1 \leq x \leq 1\}$. “x Є [-1, 1]” means that x is an element of the set [-1, 1], another way of saying x is between -1 and 1.



As the picture shows, the range of the function is the interval from 0 to 1, written in interval notation, $[0, 1]$.

Calculus -Limits and Continuity

Limits

In mathematics, the concept of a “LIMIT” is used to describe the behavior of a function as its argument or input either

1. “gets close” to some point
2. becomes arbitrarily large
3. or the behavior of a sequence's elements as their index increases indefinitely.

   Limits are used in calculus and other branches of mathematical analysis to define derivatives and continuity.

In general a limit is written like this:

```
lim  f(x) = n
x->c
```

That is to say, "as *x* gets arbitrarily close to *c*, the value of *f* goes to *n* (or gets arbitrarily close)." Take note that we **are not** concerned with what the function's value is **at** *c* but instead the value of the function as *x* gets extremely close to it.

These are the basic rules you should know. Usually you can be given an seemingly complex limit to evaluate and all you have to do is break it down into a bunch of simple limits. Let's look at some examples. First, I'm going to give an example from each rule and then I'll give you a big limit that uses all the rules. Here we go:

Rule One (no variables)

```
lim  5 = 5
x->2
```

Rule Two (plug limit value into variables)

```
lim  3x^3 - x^2 + 2x - 1 = 3(3)^3 - 3^2 + 2(3) - 1 = 77
x->3
```

Rule Three (split terms, solve separately then multiply)

```
lim  (3x+1)(2x-1) = lim  (3x+1) * lim  (2x-1) = 7 * 3 = 21
x->2                x->2          x->2
```

Rule Four (split terms, solve separately then divide)

```
                   lim  (x-1)
        x-1        x->3           2
lim    ----- =  -----------  = --- = 1/5
x->3   x^2+1    lim  (x^2+1)    10
                x->3
```

Your first order of business when dealing with limits should be to break it down to smaller limits. However, sometimes these rules are not good enough. For example:

```
lim   x^2 + 4
x->2  -------
       x - 2
```

If we were to use the methods we have learned so far we would get a division by zero. So, we must find another way. In this case you should factor out (x^2 + 4) into the multiplication of two binomials. With that done, it is sometimes possible to cancel a few terms out a get away from dividing from zero. Let's use the example from above:

```
lim   x^2 + 4     (x-2)(x+2)
x->2  -------  =  ----------
       x - 2         x - 2
```

the (x-2)'s in the numerator and denominator cancel out and you are left with:

```
lim  x + 2 = 4
x->2
```

Continuity

Enough with computing limits. I believe you have enough tools for finding the limits of pretty much any function so we will go on to continuity and then we will be done. CONTINUITY is a property functions can have which determines if a function is smooth or not. Simply put, a function is continuous if you could draw it on a piece of graph paper without having to lift your pencil.

An entire function is continuous if the function is continuous at every point in its domain. A function is continuous at one point if the one-sided limit at that point is equal to the value of the function at that point. In general:

```
If:

   lim  f(x) = f(c)
   x->c

Then the function is continuous at point [c, f(c)]
```

Now the only thing keeping us back from checking if a point on a graph is continuous is evaluating the one-sided limit. First you must evaluate the left and right-side limits. If those two limits equal each other then that is what the one-sided limit is. **If the two limits are not equal to each other then the one-sided limit does not exist.** So, if we were to rewrite the above to give a better definition:

If:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

(c- is the function coming from the left side. C+ is the function from the right side)

Then the function is continuous at point [c, f(c)]

Most functions are continuous. The only ones you usually need to worry about being non-continuous, are **piece-wise** defined functions. (ugly looking functions that usually contain several things in its domain)

Before we try a few noncontinuous functions out lets look at some continuous ones.

Always Continuous

Polynomials are continuous by definition.

Rational functions are continuous in all of its domain except when the denominator is zero. Trigonometric functions are also continuous except at points where the domain is undefined [for

example: tan (pi/2)]. Also, in addition to those functions, the sum, difference, product, and quotient of any two (or more)

Continuous functions are continuous at all parts of the domain except when the denominator is zero.

Those functions we just reviewed cover pretty much every type of function there is.

Checking Continuity

The functions that will give you trouble are piecewise-defined functions. Let's say we were give the following function:

$$f(x) = \begin{cases} x^2 + 3 \text{ for } x < 0 \\ -x^2 + 3 \text{ for } x \geq 0 \end{cases}$$

And we were asked to check if the function was continuous at (x = 0). The first thing we should do is to evaluate both two sided limits:

$$\lim_{x \to 0^-} f(x) = 0^2 + 3 = 3$$

$$\lim_{x \to 0^+} f(x) = -0^2 + 3 = 3$$

So, we have found that the limit of the function, coming from both sides of (x = 0), are equal to each other. This does not necessarily mean that the function is continuous at (x = 0) though. Thus, after you check if both the two-sided limits are equal you must then check if they are equal to the value of that function at that point:

$$\lim_{x \to 0-} f(x) = 3$$

$$\lim_{x \to 0+} f(x) = 3 \quad \text{(from above)}$$

Therefore the function at that point inherits all the attributes of being continuous. It has the same two-sided limit at the point and the limits equal the value of the function at the point. Also, since the piecewise-defined function is made up of two continuous functions the entire function is continuous. Before getting too comfortable with the notion of continuity take into consideration that the above problem is rather trivial. The problems can get more difficult. But, as long as you can compute limits you should have no problem.

The end ... and for those who feel lost ...

For some reason I feel I may have lost a few people along the way. Although I believe that if you read over the tutorial again and do examples you will probably get it let me just tell you that limits are not an absolute necessity for understanding derivatives. So do not feel bad if some of

this went over your head because derivatives are the important things. In fact, I have read some **very** good calculus books that never even mentioned the word "limit".

Here are a few useful examples of Limits and Continuity

find the value of the limits (sorry about the spacing, it might be easier to write it out to visualize the problem better)

1. lim (x+4)
 x->-1

2. lim (sqrt(x+1))/(x+2)
 x->-2

3 lim (x^2+4x-5)/(x+5)
x->-5

4 lim (-5)
x->2

Answers

1. 3
2. -4
3. -6
4. -5

What is Differentiation?

To put it simply, differentiation is the act of computing a derivative. But maybe defining 'derivative' would be more helpful. The derivative of a function lets you find another function that relates the **rate of change** of one variable to another. When I first got introduced to the concept of “rate of change” I freaked out because I didn't know what it meant-- don't freak out. It gets much simpler the longer you apply the concept to actual problems.

For example, imagine you're observing the distance a car travels from one point to another. We will arbitrarily name the position of the car *x*. It's fair to say that the position of the car, *x*, will change as time goes on. Therefore, *x*, the position, is dependent on *t*, which is time. A way of re-phrasing this in mathematical terms is saying that the position *x* = *x(t),* or, time dependent. You can't determine the position without taking time into account. Let's gather our information now. We have a way of finding out where the car is at a specific time because *x* is dependent on *t*. However, we still need a way to find how the car's position changes as time goes on. This is where differentiation comes in. Differentiation will let you observe *dx/dt*, which is the mathematical way of expressing how the car's position changes with time.

Differentiating a function can also be called finding the **derivative** of the function. There are several ways to look at this. First and probably most importantly, the slope of a tangent line on a function is the function's derivative. Let's look at what a slope and tangent line are.

Slope

The slope of a line is a way to measure the incline of the line. For example, a straight line has a slope of zero. A line from the bottom left of an axes to the top right of the axes has a positive slope, whereas a line that goes from the top left to the bottom right has a negative slope.

One way to define the slope of a line is by determining how much the line goes a distance *y* over a horizontal motion *x*. We call this the "change in *y* over the change in *x*." We can write this in mathematical terms like this:

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

In theory, this idea seems really confusing and abstract, but it is much easier when you use it with actual numbers. Let me give you an example. Pretend you have two points on the same line. The first point is ***P*** with the coordinates (***x1, y1***) and the second point is ***Q*** which has the coordinates (***x2, y2***). To find the change in *x* distance from ***P*** to ***Q***, you can use this formula:

the change in *x* from P to Q equals x2 minus x1

$$\Delta x = x_2 - x_1$$

the change in *y* from P to Q equals y2 minus y1

$$\Delta y = y_2 - y_1$$

The complete formula for the slope of a line between points P and Q is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Here's an example with numbers:

Let's say that your point P is (2,4) and point Q is (1,6).

The slope of the line between the points P and Q is (6 – 4) / (1 – 2). The result is -2.

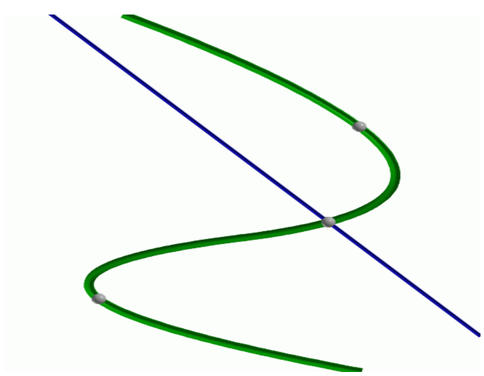
Alternatively, we can define slope trigonometrically, using the tangent function:

$$\text{Slope} = \tan(\alpha)\,,$$

(where α is the angle from the line to the rightward-pointing horizontal (measured clockwise). If you recall that the tangent of an angle in a right triangle is defined as the length of the side opposite the angle over the length of the leg adjacent to the angle, you should be able to spot the equivalence here.

The slope of a curve

Now, finding the slope of a curved line is different than finding that of a straight one. This is when we discuss the idea of ***tangents***. A tangent is a straight line that only touches a curve at one point on the graph so that the angle between the curve and the line is zero. Here is an example of a line that is **not** tangent to the curve at one point.

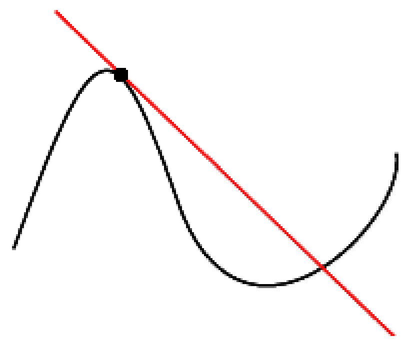


The gray dot in the middle represents the point at which the tangent intersects the curve. As you can see, the tangent line does not touch the line at only one point, nor is the angle of intersection zero. Instead, it crosses through the curve.

Here is an example of a tangent line:

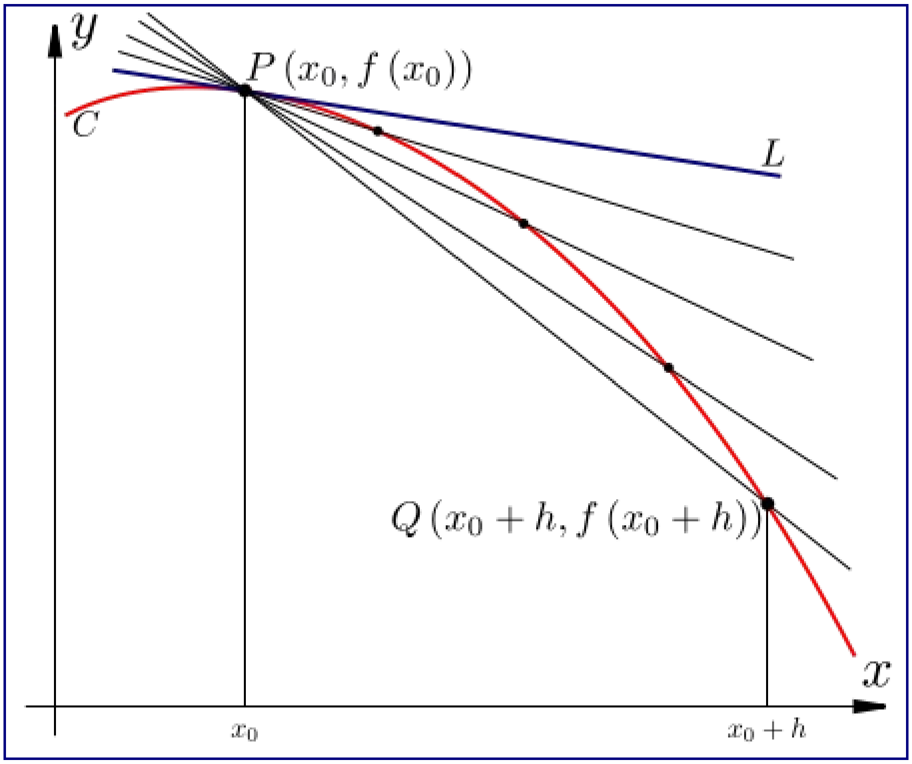
curve= **C**

The black dot indicates the point at which the red line is tangent to **C**.



Even though the tangent line crosses the curve at two points, it is tangent to **C** only at the black point.

The Slope of a Function



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting in the points on the line, (We do this in order to understand slope in function notation.)

$$m_h = \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0}.$$

This simplifies to

$m_h = \dfrac{f(x_0 + h) - f(x_0)}{h}$. which equals, (change in x)/(change in y) = (Δx)/(Δy)

Note: that H can be negative and *cannot* be zero because this would cause a division by zero.

This formula gives the average velocity over a period of time, but suppose we want to define the instantaneous velocity. In order to do this, we look at the **change in position as the change in time approaches 0**. Mathematically this is written as: $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$, which we abbreviate by the symbol $\frac{dx}{dt}$. (The idea of this notation is that the letter *d* denotes change.) Compare the symbol *d* with Δ. The (entirely non-rigorous) idea is that both indicate a difference between two numbers,

but Δ denotes a finite difference while *d* denotes an infinitesimal, or immeasurably small, difference.

To get the slope at any point for the function y=f(x), we simply follow the $\frac{dx}{dt}$ formula. However, it is useful to be able to graph all of the slopes of all the tangents lines over a certain interval. Or, put in other words, imagine that you have to find the slope of the tangent line to f(1), f(2), f(3), f(4), etc all the way through to f(50). We could calculate all of the slopes by hand. However, we prefer to create a function that calculates these slopes for us.

We call this process (of computing $\frac{dy}{dx}$) **differentiation**. Differentiation results in another function whose value for any value x is the slope of the original function at x. This function is known as the **derivative** of the original function.

Since lots of different sorts of people use derivatives, there are lots of different mathematical notations for them. Here are some:

- $f'(x)$ (read "f prime of x") for the derivative of $f(x)$,
- $D_x[f(x)]$,
- $Df(x)$,
- $\frac{dy}{dx}$ for the derivative of y as a function of x or
- $\frac{d}{dx}[y]$, which is more useful in some cases.

Most of the time the brackets are not needed, but are useful for clarity if we are dealing with something like $D(fg)$, where we want to differentiate the product of two functions, f and g.

The first notation has the advantage that it makes clear that the derivative is a function. That is, if we want to talk about the derivative of $f(x)$ at $x = 2$, we can just write $f'(2)$.

In any event, here is the formal definition:

Definition: (derivative)
Let $f(x)$ be a function. Then $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ wherever this limit exists. In this case we say that f is **differentiable** at x and its **derivative** at x is $f'(x)$.

Examples

1. The derivative of $f(x) = x / 2$ is

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{\frac{x+\Delta x}{2} - \frac{x}{2}}{\Delta x} \right) = \lim_{\Delta x \to 0} \left(\frac{1}{2} \right) = \frac{1}{2},$$

no matter what *x* is. This is consistent with the definition of the derivative as the slope of a function.

2. What is the slope of the graph of $y = 3x^2$ at (4,48)? We can do it "the hard (and imprecise) way", *without* using differentiation, as follows, using a calculator and using small differences below and above the given point:

When $x = 3.999$, $y = 47.976003$.

When $x = 4.001$, $y = 48.024003$.

Then the difference between the two values of x is $\Delta x = 0.002$.

Then the difference between the two values of y is $\Delta y = 0.048$.

Thus, the slope $= \frac{\Delta y}{\Delta x} = 24$ at the point of the graph at which $x = 4$.

But, to solve the problem precisely, we compute

$$\lim_{\Delta x \to 0} \frac{3(4+\Delta x)^2 - 48}{\Delta x} = 3 \lim_{\Delta x \to 0} \frac{(4+\Delta x)^2 - 16}{\Delta x}$$

$$= 3 \lim_{\Delta x \to 0} \frac{16 + 8\Delta x + (\Delta x)^2 - 16}{\Delta x}$$

$$= 3 \lim_{\Delta x \to 0} \frac{8\Delta x + (\Delta x)^2}{\Delta x}$$

$$= 3 \lim_{\Delta x \to 0} (8 + \Delta x)$$

$= 3(8)$

$= 24.$

We were lucky this time; the approximation we got above turned out to be exactly right. But this won't always be so, and, anyway, this way we didn't need a calculator.

In general, the derivative of $f(x) = 3x^2$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - 3x^2}{\Delta x}$$

$= 3 \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$

$= 3 \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$

$= 3 \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$

$= 3 \lim_{\Delta x \to 0} (2x + \Delta x)$

$= 3(2x)$

$= 6x$.

3. If $f(x) = |x|$ (the absolute value function) then $f'(x) = \begin{cases} -1, & x < 0 \\ \text{undefined}, & x = 0 \\ 1, & x > 0 \end{cases}$.

Here, $f(x)$ is not smooth (though it is continuous) at $x = 0$ and so the limits $\lim_{x \to 0^+} f'(x)$ and $\lim_{x \to 0^-} f'(x)$ (the limits as 0 is approached from the right and left respectively) are not equal.

From the definition, $f'(0) = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$, which does not exist. Thus, $f'(0)$is undefined, and so $f'(x)$ has a discontinuity at 0. This sort of point of non-differentiability is called a cusp. Functions may also not be differentiable because they go to infinity at a point, or oscillate infinitely frequently.

Calculus/Newton's Method

Newton's Method (also called the Newton-Raphson method) is a recursive algorithm for approximating the root of a differentiable function. We know simple formulas for finding the roots of linear and quadratic equations, and there are also more complicated formula for cubic and quartic equations. At one time it was hoped that there would be formulas found for equations of quintic and higher-degree, though it was later shown by Neils Henrik Abel that no such equations exist. The Newton-Raphson method is a method for approximating the roots of polynomial equations of any order. In fact the method works for any equation, polynomial or not, as long as the function is differentiable in a desired interval.

Newton's Method

Let $f(x)$ be a differentiable function. Select a point x_1 based on a first approximation to the root, arbitrarily close to the function's root. To approximate the root you then recursively calculate using:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

As you recursively calculate, the x_n's become increasingly better approximations of the function's root.
For n number of approximations,

$$x_n = x_0 - \sum_{i=0}^{n} \frac{f(x_i)}{f'(x_i)}$$

In order to explain Newton's method, imagine that x_0 is already very close to a zero of $f(x)$. We know that if we only look at points very close to x_0 then $f(x)$ looks like it's tangent line. If x_0 was already close to the place where $f(x)$ was zero, and near x_0 we know that $f(x)$ looks like its tangent line, then we hope the zero of the tangent line at x_0 is a better approximation then x_0 itself.

The equation for the tangent line to $f(x)$ at x_0 is given by

$$y = f'(x_0)(x - x_0) + f(x_0).$$

Now we set $y = 0$ and solve for x.

$$0 = f'(x_0)(x - x_0) + f(x_0)$$

$$-f(x_0) = f'(x_0)(x - x_0)$$

$$\frac{-f(x_0)}{f'(x_0)} = (x - x_0)$$

$$x = \frac{-f(x_0)}{f'(x_0)} + x_0$$

This value of *x* we feel should be a better guess for the value of *x* where *f*(*x*) = 0. We choose to call this value of x_1, and a little algebra we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If our intuition was correct and x_1 is in fact a better approximation for the root of *f*(*x*), then our logic should apply equally well at x_1. We could look to the place where the tangent line at x_1 is zero. We call x_2, following the algebra above we arrive at the formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

And we can continue in this way as long as we wish. At each step, if your current approximation is x_n our new approximation will be $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$

Examples

Find the root of the function $f(x) = x^2$.

$$x_1 = f(2) = 4$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{1}{2}$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = \frac{1}{4}$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = \frac{1}{8}$$

$$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)} = \frac{1}{16}$$

$$x_8 = x_7 - \frac{f(x_7)}{f'(x_7)} = \frac{1}{32}$$

As you can see x_n is gradually approaching zero (which we know is the root of *f(x)*). One can approach the function's root with arbitrary accuracy.

`Answer:` $f(x) = x^2$ `has a root at` $x = 0$.

Notes

This method fails when $f'(x) = 0$. In that case, one should choose a new starting place. Occasionally it may happen that $f(x) = 0$ and $f'(x) = 0$ have a common root. To detect whether this is true, we should first find the solutions of $f'(x) = 0$, and then check the value of $f(x)$ at these places.

Newton's method also may not converge for every function, take as an example:

$$f(x) = \begin{cases} \sqrt{x-r}, & \text{for} x \geq r \\ -\sqrt{r-x}, & \text{for} x \leq r \end{cases}$$

For this function choosing any $x_1 = r - h$ then $x_2 = r + h$ would cause successive approximations to alternate back and forth, so no amount of iteration would get us any closer to the root than our first guess.

Calculus/Related Rates

Introduction

Process for solving related rates problems:

- Write out any relevant formulas and information.
- Take the derivative of the primary equation with respect to time.

- Solve for the desired variable.
- Plug-in known information and simplify.

Related Rates

As stated in the introduction, when doing related rates, you generate a function which compares the rate of change of one value with respect to change in time. For example, velocity is the rate of change of distance over time. Likewise, acceleration is the rate of change of velocity over time. Therefore, for the variables for distance, velocity, and acceleration, respectively x, v, and a, and time, t:

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

Using derivatives, you can find the functions for velocity and acceleration from the distance function. This is the basic idea behind related rates: the rate of change of a function is the derivative of that function with respect to time.

Common Applications

Filling Tank

This is the easiest variant of the most common textbook related rates problem: the filling water tank.

- The tank is a cube, with volume 1000L.
- You have to fill the tank in ten minutes or you die.
- You want to escape with your life and as much money as possible, so you want to find the smallest pump that can finish the task.

We need a pump that will fill the tank 1000L in ten minutes. So, for pump rate p, volume of water pumped v, and minutes t:

$$p = \frac{dv}{dt}$$

Examples

Related rates can get complicated very easily.

Example 1:

```
A cone with a circular base is being filled with water. Find a formula which
will find the rate with which water is pumped.
```

- Write out any relevant formulas or pieces of information.

$$V = \frac{1}{3}\pi r^2 h$$

- Take the derivative of the equation above with respect to time. Remember to use the Chain Rule and the Product Rule.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2rh \cdot \frac{dr}{dt}\right)$$

Answer: $\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \cdot \frac{dh}{dt} + 2rh \cdot \frac{dr}{dt}\right)$

Example 2:

A spherical hot air balloon is being filled with air. The volume is changing at a rate of 2 cubic feet per minute.
How is the radius changing with respect to time when the radius is equal to 2 feet?

- Write out any relevant formulas and pieces of information.

$$V_{sphere} = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 2$$

$$r = 2$$

- Take the derivative of both sides of the volume equation with respect to time.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3} \cdot 3 \cdot \pi r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

- Solve for $\frac{dr}{dt}$.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}$$

- Plug-in known information.

$$\frac{dr}{dt} = \frac{1}{16\pi} \cdot 2$$

Answer: $\frac{dr}{dt} = \frac{1}{8\pi}$ ft/min.

Example 3:

An airplane is attempting to drop a box onto a house. The house is 300 feet away in horizontal distance and 400 feet in vertical distance. The rate of change of the horizontal distance with respect to time is the same as the rate of change of the vertical distance with respect to time. How is the distance between the box and the house changing with respect to time at the moment? The rate of change in the horizontal direction with respect to time is -50 feet per second.

Note: Because the vertical distance is downward in nature, the rate of change of y is negative. Similarly, the horizontal distance is decreasing, therefore it is negative (it is getting closer and closer).

The easiest way to describe the horizontal and vertical relationships of the plane's motion is the Pythagorean Theorem.

- Write out any relevant formulas and pieces of information.

$x^2 + y^2 = s^2$ (where *s* is the distance between the plane and the house)

$$x = 300$$

$$y = 400$$

$$s = \sqrt{x^2 + y^2} = \sqrt{300^2 + 400^2} = 500$$

$$\frac{dx}{dt} = \frac{dy}{dt} = -50$$

- Take the derivative of both sides of the distance formula with respect to time.

$$x^2 + y^2 = s^2$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2s \cdot \frac{ds}{dt}$$

- Solve for $\frac{ds}{dt}$.

$$\frac{ds}{dt} = \frac{1}{2s}\left(2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}\right)$$

- Plug-in known information

$$\frac{ds}{dt} = \frac{1}{2(500)}[2(300) \cdot (-50) + 2(400) \cdot (-50)]$$

$$= \frac{1}{1000}(-70000)$$

$$= -70 \text{ ft/s}$$

Answer: $\frac{ds}{dt} = -70$ ft/sec.

Example 4:

```
Sand falls onto a cone shaped pile at a rate of 10 cubic feet per minute.
The radius of the pile's base
is always 1/2 of its altitude.  When the pile is 5 ft deep, how fast is the
altitude of the pile increasing?
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- Write down any relevant formulas and information.

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = 10$$

$$r = \frac{1}{2}h$$

$$h = 5$$

Substitute $r = \frac{1}{2}h$ into the volume equation.

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi h \cdot \left(\frac{h^2}{4}\right)$$

$$= \frac{1}{12}\pi h^3$$

- Take the derivative of the volume equation with respect to time.

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \cdot \frac{dh}{dt}$$

- Solve for $\frac{dh}{dt}$.

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

- Plug-in known information and simplify.

$$\frac{dh}{dt} = \frac{4}{\pi(5)^2} \cdot 10$$

$$= \frac{8}{5\pi} \text{ ft/min}$$

Answer: $\frac{dh}{dt} = \frac{8}{5\pi}$ ft/min.

Example 5:

A 10 ft long ladder is leaning against a vertical wall. The foot of the ladder is being pulled away from the wall at a constant rate of 2 ft/sec. When the ladder is exactly 8 ft from the wall, how fast is

the top of the ladder sliding down the wall?

- Write out any relevant formulas and information.

Use the Pythagorean Theorem to describe the motion of the ladder.

$x^2 + y^2 = l^2$ (where *l* is the length of the ladder)

$$l = 10$$

$$\frac{dx}{dt} = 2$$

$$x = 8$$

$$y = \sqrt{l^2 - x^2} = \sqrt{100 - 64} = \sqrt{36} = 6$$

- Take the derivative of the equation with respect to time.

$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$ (l^2 so $\frac{dl}{dt} = 0$.)

- Solve for $\frac{dy}{dt}$.

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$2y \cdot \frac{dy}{dt} = -2x \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

- Plug-in known information and simplify.

$$\frac{dy}{dt} = \left(-\frac{8}{6}\right)(-2)$$

$= \frac{8}{3}$ ft/sec

Answer: $\frac{dy}{dt} = \frac{8}{3}$ ft/sec.

Calculus/Optimization

Introduction

Optimization is the use of Calculus in the real world. Let us assume we are a pizza parlor and wish to maximize profit. Perhaps we have a flat piece of cardboard and we need to make a box with the greatest volume. How does one go about this process?

Obviously, this requires the use of maximums and minimums. We know that we find maximums and minimums via derivatives. Therefore, one can conclude that Calculus will be a useful tool for maximizing or minimizing (also known as "Optimizing") a situation.

Examples

Volume Example

A box manufacturer desires to create a box with a surface area of 100 inches squared. What is the maximum size volume that can be formed by bending this material into a box? The box is to be closed. The box is to have a square base, square top, and rectangular sides.

- Write out known formulas and information

$$A_{base} = x^2$$

$$A_{side} = x \cdot h$$

$$A_{total} = 2x^2 + 4x \cdot h = 100$$

$$V = l \cdot w \cdot h = x^2 \cdot h$$

- Eliminate the variable *h* in the volume equation

$$2x^2 + 4xh = 100$$

$$x^2 + 2xh = 50$$

$$2xh = 50 - x^2$$

$$h = \frac{50 - x^2}{2x}$$

$$V = (x^2)\left(\frac{50 - x^2}{2x}\right)$$

$$= \frac{1}{2}(50x - x^3)$$

- Find the derivative of the volume equation in order to maximize the volume

$$\frac{dV}{dx} = \frac{1}{2}(50 - 3x^2)$$

- Set $\frac{dV}{dx} = 0$ and solve for x

$$\frac{1}{2}(50 - 3x^2) = 0$$

$$50 - 3x^2 = 0$$

$$3x^2 = 50$$

$$x = \pm\frac{\sqrt{50}}{\sqrt{3}}$$

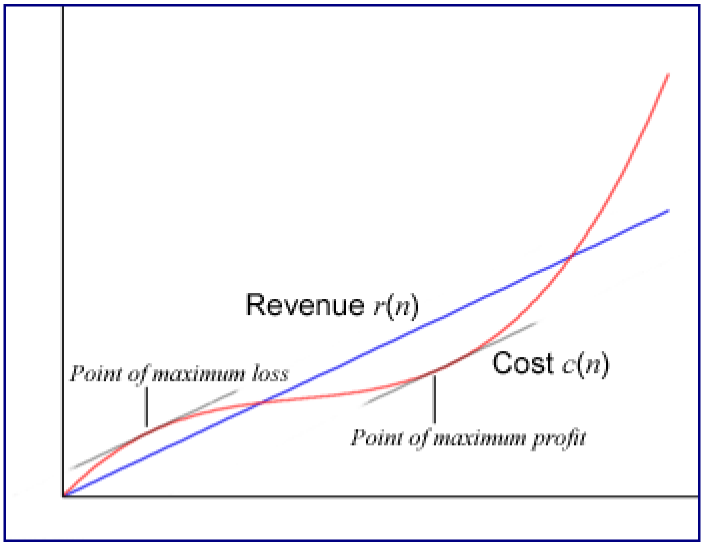
- Plug-in the *x* value into the volume equation and simplify

$$V = \frac{1}{2}\left[50 \cdot \sqrt{\frac{50}{3}} - \left(\sqrt{\frac{50}{3}}\right)^3\right]$$

$$= 68.04138174..$$

`Answer:` $V_{max} = 68.04138174..$

Sales Example



A small retailer can sell n units of a product for a revenue of $r(n) = 8.1n$ and at a cost of $c(n) = n^3 - 7n^2 + 18n$, with all amounts in thousands. How many units does it sell to maximize its profit?

The retailer's profit is defined by the equation $p(n) = r(n) - c(n)$, which is the revenue generated less the cost. The question asks for the maximum amount of profit which is the maximum of the above equation. As previously discussed, the maxima and minima of a graph are found when the slope of said graph is equal to zero. To find the slope one finds the derivative of $p(n)$. By using the subtraction rule $p'(n) = r'(n) - c'(n)$:

$$p(n) = r(n) - c(n)$$

$$p'(n) = \frac{d}{dn}[8.1n] - \frac{d}{dn}\left[n^3 - 7n^2 + 18n\right]$$

$$= -3n^2 + 14n - 9.9$$

Therefore, when $-3n^2 + 14n - 9.9 = 0$ the profit will be maximized or minimized. Use the quadratic formula to find the roots, giving $\{3.798, 0.869\}$. To find which of these is the maximum and minimum the function can be tested:

$p(0.869) = -3.97321, p(3.798) = 8.58802$

Because we only consider the functions for all $n \geq 0$ (i.e., you can't have $n = -5$ units), the only points that can be minima or maxima are those two listed above. To show that 3.798 is in fact a maximum (and that the function doesn't remain constant past this point) check if the sign of $p'(n)$ changes at this point. It does, and for n greater than 3.798 $P'(n)$ the value will remain decreasing. Finally, this shows that for this retailer selling 3,798 units would return a profit of \$8,588.02.